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# Measurement of the velocity of a Dirac particle

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**Abstract.** Using a model quantum clock, I show how the velocity of a relativistic particle can be measured. The results are used to analyse the long-standing problem of the velocity of a Dirac particle.

## 1. Background

The problem of the velocity of a Dirac particle has a long history (see, for example, Dirac 1947). It is usually discussed in connection with the so-called 'zitterbewegung' phenomenon. Interest in the subject arises because of the existence of an explicit velocity operator in Dirac theory, which commutes with both the position and momentum operators. The eigenvalues of the velocity operator in a particular direction are $\pm 1$ (we use units $\hbar = c = 1$). It would therefore appear that Dirac particles, even with a non-zero rest mass, move only with the speed of light. This situation does not arise for non-Dirac particles, for which the velocity can take continuous values in the range $[-1, 1]$.

Dirac (1947) explained this curious result as follows. Measurement of velocity requires the accurate measurement of position of the particle at subsequent times. The first position measurement will produce a sharply peaked wavepacket and introduce a large uncertainty in momentum. Hence the particle will almost certainly be found subsequently with a very large momentum and a velocity close to the speed of light.

Dirac's argument has been criticised by Loba (1956) and Aharonov and Bohm (1957), who investigated the behaviour of Dirac wavepackets and concluded that there was no difference in the behaviour of highly localised Dirac and non-Dirac particles. Thus the novel features of Dirac particles with regard to velocity are not made explicit in the measurement strategy suggested by Dirac.

Aharonov and Bohm (1957), in assessing this negative evidence for the peculiar nature of the velocity of Dirac particles, pointed out that a more promising approach might be to investigate the measurement of velocity for a particle with well defined momentum, rather than position (as they had considered). In that case one might be able to exhibit a situation in which the expectation value of momentum was arbitrarily small but the velocity was that of light. One could then envisage a beam of massive particles moving at the speed of light exerting negligible pressure! But the authors went on to counter that a momentum eigenstate would correspond to an infinitely extended plane wave for which the concept of two position measurements is meaningless.

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However, in this objection they are wrong, as has been shown by Peres (1980). If one merely requires the time *interval* taken by a particle to travel between two fixed points, and not the actual moment of passage, there is no need to localise the particle at all.

Following Peres, I here construct a model quantum clock that only runs when a particle is passing between two points a fixed distance apart. By reading the clock at $t=+\infty$ the duration of passage $T$, and hence the particle's velocity $v=L/T$, is measured. This system can then be used to investigate the velocity of particles in momentum eigenstates, and to compare the results for Dirac, Klein-Gordon and Schrödinger particles.

The Dirac equation for a particle of rest mass $m_0$ may be written

$$(\mathrm{i}\,\partial/\partial t+\mathrm{i}\boldsymbol{\alpha}\cdot\nabla-\beta m_0)\psi=0 \tag{1}$$

where

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix} \qquad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$$

$\sigma_x$, $\sigma_y$, $\sigma_z$ are Pauli spin matrices and $I$ is a unit $2\times2$ matrix. In this form $\boldsymbol{\alpha}$ can be interpreted as a velocity operator; for example

$$\mathrm{d}z/\mathrm{d}t=\alpha_z. \tag{2}$$

## 2. The model clock

Any measurement of the particle's velocity will require coupling between the particle and the clock. This will result in a disturbance to both the particle's motion and the normal running of the clock. To reduce this disturbance we shall consider the case of a weakly coupled (non-relativistic) quantum clock. The velocity of the particle will be measured by preparing the clock in a particular initial stationary state and then performing a subsequent observation at late times to ascertain the clock's final stationary state.

Suppose the running clock has the Hamiltonian $H_c+H'$, where $H'$ represents the driving mechanism that advances the clock through a succession of distinguishable states, and $H_c$ represents the remaining part of the clock's Hamiltonian. The clock's wavefunction can be considered to factor into the form $\Psi\psi$, where $\psi$ depends only on the degree of freedom represented by the clock's 'ticking' motion.

I shall suppose that the clock has $n$ distinct states; $n$ thus determines the resolution of the clock. Following Salecker and Wigner (1958), the clock's wavefunction is chosen to be

$$\chi(t)=n^{-1/2}\Psi\sum_{m=0}^{n-1}\psi_m\exp(-2\pi\mathrm{i}mt/n\tau) \tag{3}$$

where $\psi_m$ are stationary states:

$$H'\psi_m=(2\pi m/n\tau)\psi_m. \tag{4}$$

It is easily verified that the successive states $\chi(0),\chi(\tau),\chi(2\tau),\ldots$ are mutually orthogonal. The evolution operator $\mathrm{e}^{-\mathrm{i}H'\tau}$ advances the clock through these successive states at times $0,\tau,2\tau,\ldots$. Thus, for example,

$$\mathrm{e}^{-\mathrm{i}H'\tau}\chi(0)=\chi(\tau). \tag{5}$$

The total Hamiltonian operator for the closed system (Dirac particle + clock) is

$$H = -\mathrm{i}\boldsymbol{\alpha}\cdot\nabla + \beta m_0 + H_c + P(z)H' \tag{6}$$

where $P(z) = 1$ if $0 < z < L$ and 0 otherwise, and $z$ is the position of the particle. Thus the clock mechanism is switched on by the arrival of the particle at $z = 0$ and switched off again when it passes $z = L$. The duration of its passage from 0 to $L$ is then determined by reading how many divisions of $\tau$ the clock has passed through at late times.

## 3. Measuring the velocity

Attention will be restricted to motion in the $z$ direction only. Writing $\phi(t, z)$ for the particle wavefunction, we obtain

$$H(\phi\chi) = \mathrm{i}\,\partial/\partial t(\phi\chi). \tag{7}$$

Suppose now that the clock, rather than being in the state (3), is instead in one of the energy eigenstates: $\psi = \psi_m$. Then using (4) and separating the variables in (7) yields an energy eigenvalue equation for $\phi$:

$$(-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla + \beta m_0 + 2\pi m P(z)/n\tau)\phi = E\phi. \tag{8}$$

This is the equation for a Dirac particle moving in the square potential $V \equiv V_0 = 2\pi m/n\tau$ for $0 < z < L$, $V = 0$ elsewhere.

To solve (8) we match together solutions

$$\phi = \mathrm{e}^{\mathrm{i}kz}\begin{pmatrix}1\\0\\k/(E+m_0)\\0\end{pmatrix} + A\,\mathrm{e}^{-\mathrm{i}kz}\begin{pmatrix}1\\0\\-k/(E+m_0)\\0\end{pmatrix} \qquad z<0$$

$$B\,\mathrm{e}^{\mathrm{i}qz}\begin{pmatrix}1\\0\\q/(E-V_0+m_0)\\0\end{pmatrix} + C\,\mathrm{e}^{-\mathrm{i}qz}\begin{pmatrix}1\\0\\-q/(E-V_0+m)\\0\end{pmatrix} \qquad 0<z<L$$

$$D\,\mathrm{e}^{\mathrm{i}kz}\begin{pmatrix}1\\0\\k/(E+m_0)\\0\end{pmatrix} \qquad z>L \tag{9}$$

(ignoring an overall normalisation constant), where

$$k = (E^2 - m_0^2)^{1/2} \tag{10}$$

$$q = [(E - V_0)^2 - m_0^2]^{1/2} \tag{11}$$

with $E > 0$, $k > 0$. Continuity at $z = 0$ and $z = L$ gives four equations for the four coefficients $A$, $B$, $C$, $D$. After some work one finds

$$D = |D|\,\mathrm{e}^{\mathrm{i}\theta} \tag{12}$$

where

$$\theta = \tan^{-1}\left(\frac{(1+\beta)\sin(q-k)L + (1-\beta)^2\sin(q+k)L}{(1+\beta)\cos(q-k)L + (1-\beta)^2\cos(q+k)L}\right) \tag{13}$$

and

$$\beta = \frac{k}{q}\left(\frac{E - V_0 + m_0}{E + m_0}\right). \tag{14}$$

Thus $\theta$ is the phase shift suffered by the right-moving wave $e^{ikz}$ due to the interaction with the clock. If this interaction is small, we may neglect backscattering, so $A \simeq 0$, $|D| \simeq 1$ and use the approximations

$$q \simeq (E^2 - m_0^2)^{1/2}[1 - EV_0/(E^2 - m_0^2)] = k - (EV_0/k) \tag{15}$$

$$\beta \simeq 1 + (m_0 V_0/k^2). \tag{16}$$

This enables us to neglect the terms $(1-\beta)^2$ in (13) because they are at least quadratic in the small quantity $V_0/k$. Then

$$\theta \simeq \tan^{-1}\tan[(q-k)L] = (q-k)L \tag{17}$$

$$\simeq -EV_0L/k = -V_0L(E/p) \tag{18}$$

where $p = k$ is the particle's momentum.

Now a classical relativistic particle with energy $E$ and momentum $p$ has velocity $v = p/E$. We therefore write (18) as

$$\theta \simeq -2\pi mL/n\tau v. \tag{19}$$

The space-dependent part of the total wavefunction when the clock is in the eigenstate $\psi_m$ is therefore approximately

$$e^{ikz}\begin{pmatrix} 1 \\ 0 \\ k/(E+m_0) \\ 0 \end{pmatrix} \psi_m \qquad z<0 \tag{20}$$

$$e^{ikz}\begin{pmatrix} 1 \\ 0 \\ k/(E+m_0) \\ 0 \end{pmatrix} \psi_m \exp(-2\pi m\mathrm{i}L/n\tau v) \qquad z>L. \tag{21}$$

The superposition of eigenstates which puts the clock in the state $\chi(t)$ (see (3)) is then

$$e^{ikz}\begin{pmatrix} 1 \\ 0 \\ k/(E+m_0) \\ 0 \end{pmatrix} n^{-1/2}\sum_{m=0}^{n-1}\psi_m \qquad z<0 \tag{22}$$

and

$$e^{ikz}\begin{pmatrix} 1 \\ 0 \\ k/(E+m_0) \\ 0 \end{pmatrix} n^{-1/2}\sum_{m=0}^{n-1}\psi_m \exp(-2\pi m\mathrm{i}L/n\tau v) \qquad z>L \tag{23}$$

where we have chosen $\chi(0)$ as the initial state of the clock.

Inspection of (21) shows from (5) that the phase factor in the last exponent of (23) corresponds exactly to the effect of acting on $\chi(0)$ with the evolution operator $\exp(-iH'T)$ where $T = L/v$, i.e. the classical time of flight between the fixed points. In other words, the final reading on the clock yields the value $L/v$ to a resolution $\tau$. We assumed in (15) and (16) that $|V_0| \ll p^2/(E+m_0)$ which in turn requires $\tau \gg 2\pi/[p^2/(E+m_0)] \simeq 1/E$ for a relativistic particle. This is the expected limitation due to the Heisenberg uncertainty principle, and it imposes an uncertainty on $v$ of about

$$\Delta v \simeq v^2/L(p^2/E) = 1/LE. \tag{24}$$

I now consider the case of a relativistic particle obeying the Klein-Gordon equation. The effect of the clock is again represented by a static potential $V_0$. Thus the particle's equation of motion is

$$[(E - V_0)^2 + \nabla^2 - m_0^2]\phi = 0. \tag{25}$$

An identical argument to the foregoing establishes that $V_0 = 2\pi m/n\tau$, in the interval $0 < x < L$. Choosing incoming waves of the form $\phi = e^{ikz}$, and matching $\phi$ and $\partial\phi/\partial z$ at $z = 0$ and $z = L$, again leads to an equation for the phase shift in the wave due to the potential $V_0$. This time one obtains

$$\theta = \tan^{-1}\left(\frac{\tan qL - [2kq/(q^2+k^2)] \tan kL}{[2kq/(q^2+k^2)] + \tan qL \tan kL}\right). \tag{26}$$

The approximations (15) and (16) imply $2kq \simeq q^2 + k^2$, so (26) gives

$$\theta \simeq \tan^{-1}[\tan(q-k)L] = (q-k)L = -2\pi mL/n\tau v \tag{27}$$

exactly as before. In particular, $T = L/v$, and we are led to the same uncertainty relation (24).

In the non-relativistic limit, (24) reduces to

$$\Delta v \simeq 1/m_0 L \tag{28}$$

which is the result found by Peres (1980) for a Schrödinger particle.

## 4. Discussion

We are now in a position to compare the status of velocity measurements for the three cases of Dirac, Klein-Gordon and Schrödinger particles in momentum eigenstates. For the state chosen,

$$e^{ikz}\begin{pmatrix} 1 \\ 0 \\ k/(E+m_0) \\ 0 \end{pmatrix} \tag{29}$$

the measured velocity can take any values between 0 and 1. Note that the replacement $k \to -k$ (and $q \to -q$) reverses the sign of the phase shift $\theta$ in (17) and yields a value for $v$ in the range $-1$ to 0. This corresponds to a left-moving wave.

The situation is no different from the Klein-Gordon case, where the same continuous set of values for $v$ is obtained. Both cases reduce in the non-relativistic limit, as expected, to the Schrödinger case investigated by Peres (1980). Indeed, although the exact expressions for the phase shift are very different in the three cases, the approximate expressions for small $V_0$ are identical.

There is, however, the question of the uncertainty $\Delta v$ given by (24). Evidently this can be made arbitrarily small by letting $L \to \infty$. In particular we can arrange for $1/LE \ll 1$ so that $v$ can be sharply defined to a value much less than the velocity of light. On the other hand, an accurate measurement of the *instantaneous* velocity for the state (29) is not possible, unless we take the infinite energy limit $E \to \infty$. In this respect there is also no difference from the non-Dirac case.

Is this result really unexpected? A measurement of velocity will only yield a velocity eigenvalue $\pm 1$ if the system is actually in a velocity eigenstate. For an arbitrary state the *expectation value* of $|v|$ will generally be less than 1. Indeed, commuting $\alpha_x$ with $H$ leads to the relation (Dirac 1947)

$$\alpha_x(t) = p_x/H + e^{2iHt}[\alpha(0) - p_x/H]. \tag{30}$$

The final term implies a rapid oscillatory motion with frequency greater than $m_0$. This is the zitterbewegung. Over time intervals much greater than the Compton time it averages to zero, leaving only the first term $\sim p_x/m_0$, which is the same as for the non-Dirac case.

Thus, to detect the peculiar velocity behaviour of the Dirac particle, it is apparently necessary to have $L < m_0^{-1}$. This involves an uncertainty in $v$ of $\Delta v \geqslant m_0/E$. For a non-relativistic particle $m_0/E \simeq 1$ and the velocity is completely indeterminate. For highly relativistic particles, however, $m_0/E \ll 1$ and $\Delta v \ll 1$. This accords with a result of Aharonov and Bohm (1957) that the zitterbewegung diminishes at high energies.

Evidently, then, a difference in the status of velocity between Dirac and Klein-Gordon particles arises in the limit $m_0 \to 0$. Inspection of (10), (11) and (14) shows that $\beta = 1$ exactly in this case, and $q - k = V_0$. Thus (13) immediately simplifies without approximation to yield in place of (18)

$$\theta = -V_0 L \tag{31}$$

without the need for small $V_0$ approximation. The energy-time uncertainty limitation (24) no longer applies, and $\Delta v = 0$. By contrast, *no* such simplification occurs in (26) for the Klein-Gordon particle, and the measurement of velocities close to that of light entails the same uncertainty, $1/LE$.

The reason for this difference concerns the fact that in the massless limit the Dirac eigenstate (29) is simultaneously an eigenstate of the momentum *and* velocity operators, something which is impossible in the Klein-Gordon case, and which arises because of the novel feature of Dirac theory that the velocity operator commutes with the momentum operator, thus implying that velocity and momentum are independent variables. The system, being in a velocity eigenstate in this limit, then permits a precise ($\Delta v = 0$) measurement of velocity.

It would be interesting to consider the measurement of velocity for velocity eigenstates with $m_0 \neq 0$. However, because $\boldsymbol{\alpha}$ only commutes with $H$ in the massless case, such velocity eigenstates would not be stationary states, and would involve superpositions of positive and negative energies. The measurement of the velocity of a particle in such states is likely to be very problematical.

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## References

Aharonov Y and Bohm D 1957 *Nuovo Cimento Suppl. Ser. X* **5** 429
Dirac P A M 1947 *Principles of Quantum Mechanics* 3rd edn (Oxford: Oxford University Press)
Koba Z 1956 *Nuovo Cimento Ser. X* **3** 1
Peres A 1980 *Am. J. Phys.* **48** 552
Salecker H and Wigner E P 1958 *Phys. Rev.* **109** 571